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A system of quantum stochastic differential equations in terms of non-equilibrium thermo field dynamics

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Abstract. Within the framework of non-equilibrium thermo field dynamics (NETFD), quantum Wiener processes at finite temperatures are constructed, and its representation space is shown to be the thermal space. After the introduction of the stochastic Shrödinger equation, a unified system of quantum stochastic differential equations, including the quantum stochastic Liouville equation and the quantum Langevin equation, is established within the quantum stochastic calculus.

1. Introduction

Studies of the Langevin equation for quantum systems were started by Senitzky [1], Lax [2] and Haken [3]. They investigated the Langevin equation for a quantum mechanical damped harmonic oscillator. In the quantum Langevin equation, variables in both relevant and irrelevant systems are stochastic operators. Putting the condition that the equal-time canonical commutation relation should hold for all time even for stochastic operators, they derived commutation relations among random force operators and their correlations.

In their studies, Senitzky, Lax and Haken did not construct a representation space explicitly. In quantum theory, observable operators do not have physical meaning until a representation space is specified. As was pointed out by Kubo [4], the quantum Langevin equation is an operator equation defined on a total representation space, i.e. a space of a relevant system and of random forces. Any representation space of random force operators had not been constructed by physicists.

Mathematicians such as Hudson, Parthasarathy and their co-workers [5–10] constructed explicitly a representation space of random force operators. With the representation space, they realized a stochastic Schrödinger equation by analogy with the usual quantum mechanics. A time-evolution generator satisfying the stochastic Schrödinger equation was determined on the requirement of its unitarity, which is one of the necessary conditions for construction of a canonical operator formalism. It seems that, for mathematicians, a construction of the stochastic Liouville equation was out of their considerations.

The stochastic Liouville equation was introduced first by Kubo and co-workers [11, 12] in order to investigate classical stochastic systems. In classical systems, the stochastic Liouville equation is an equation of motion for a probability distribution function in phase space under the influence of random forces. There had been a few attempts to extend the stochastic Liouville equation to quantum systems. Parkins–Gardiner [13, 14] and Dekker [15] derived a quantum stochastic Liouville equation by obtaining, within the trace formalism, an adjoint operator of a time-evolution generator for the quantum

Langevin equation. Furthermore, Gardiner *et al* [16] rederived their stochastic Liouville equation on the basis of the stochastic Schrödinger equation introduced by Hudson *et al* by making use of the fact that a density operator is a functional of wavefunctions. Within the density operator formalism, it is impossible to extract an explicit form of the time-evolution generator satisfying the stochastic Liouville equation, since the Liouville equation has entanglements between relevant operators and a density operator due to commutators and anticommutators among them. These difficulties prevent one from constructing a canonical operator formalism based on the stochastic Liouville equation.

On the other hand, within the framework of non-equilibrium thermo field dynamics (NETFD) [17–21], a unified canonical operator formalism of quantum stochastic differential equations was constructed [22–32] on the basis of the stochastic Liouville equation. The quantum stochastic differential equations include the quantum Langevin equation and the quantum stochastic Liouville equation together with the corresponding quantum master equation. Within NETFD, introducing two kinds of operators, *with* tilde and *without* tilde, the entanglements between relevant operators and a density operator in the stochastic Liouville equation can be disentangled. Therefore, one can extract the explicit form of the time-evolution generator satisfying the stochastic Liouville equation, which enables us to construct a unified canonical operator formalism.

In this paper, we will construct quantum Wiener processes by means of boson annihilation and creation operators with their representation space extending mathematicians' procedure and implanting it into NETFD (section 2). The thermal degree of freedom in the quantum Wiener processes will be introduced by a *Bogoliubov transformation* in the thermal space which is a representation space within NETFD. On the basis of the quantum Wiener processes, we will establish a quantum stochastic calculus (section 3). Requiring the unitarity of the time evolution of the stochastic wavefunction, we will construct a stochastic Schrödinger equation (section 4). Then, starting from the stochastic Schrödinger equation, we will show how one can obtain the time-evolution generator satisfying a stochastic Liouville equation with the help of the fact that a density operator is a functional of wavefunctions together with the principle of correspondence between quantities in the thermal space and in the Hilbert space (section 5). We will also show how one can construct a unified canonical operator formalism of quantum stochastic differential equations on the basis of the time-evolution generator (section 5).

2. Quantum Wiener processes

We will construct quantum Wiener processes at zero temperature according to Hudson and Parthasarathy [5, 9, 10].

2.1. Fock Space

We introduce boson operators $b(t)$ and $b^\dagger(t)$ with $t \in [0, \infty)$ satisfying the canonical commutation relations

$$[b(t),\ b^\dagger(t)] = \delta(t-s) \qquad [b(t),\ b(s)] = 0 \tag{1}$$

and define the vacuums $|0\rangle\rangle$ and $\langle\langle 0|$ by

$$b(t)|0\rangle\rangle = 0 \qquad \langle\langle 0|b^\dagger(t) = 0. \tag{2}$$

We introduce ket- and bra-vectors defined by

$$|t_1, \ldots, t_n\rangle\rangle = \frac{1}{\sqrt{n!}} b^\dagger(t_1) \ldots b^\dagger(i_n)|0\rangle\rangle \qquad \langle\langle t_1, \ldots, t_n| = \langle\langle 0| \frac{1}{\sqrt{n!}} b(t_1) \ldots b(t_n) \tag{3}$$

which satisfy the orthonormalization condition

$$\langle\langle t_1, \ldots, t_n | s_1, \ldots, s_m \rangle\rangle = \delta_{nm} \frac{1}{n!} \sum_P \delta(t_1 - s_1) \ldots \delta(t_n - s_n) \tag{4}$$

and the completeness relation

$$\sum_{n=0}^{\infty} \left(\prod_l^n \int_0^\infty \mathrm{d}t_l \right) | t_1, \ldots, t_n \rangle\rangle \langle\langle t_1, \ldots, t_n | = I. \tag{5}$$

Here, $\sum_P$ indicates the summation over all possible permutations of $t_1, \ldots, t_n$ with $s_1, \ldots, s_n$ fixed. Therefore, the set of ket-vectors $\{|t_1, \ldots, t_n\rangle\rangle\}$ and that of bra-vectors $\{\langle\langle t_1, \ldots, t_n|\}$ form complete orthonormal systems. The vector space Γ^0 spanned by the complete orthonormal basic vectors $|t_1, \ldots, t_n\rangle\rangle$ and $\langle\langle t_1, \ldots, t_n|$ is called the *Fock space*†.

2.2. Quantum Wiener processes

Let us define the operators B_t and $B_t^\dagger$ on the Fock space Γ^0 by

$$B_t = \int_0^t \mathrm{d}s\, b(s) \qquad B_t^\dagger = \int_0^t \mathrm{d}s\, b^\dagger(s). \tag{6}$$

Taking expectations of B_t, $B_t^\dagger$ and the product $B_t^\dagger B_s$, $B_t B_s^\dagger$ with respect to the vacuums $|0\rangle\rangle$ and $\langle\langle 0|$, we find that

$$\langle\langle 0|B_t|0\rangle\rangle = \langle\langle 0|B_t^\dagger|0\rangle\rangle = 0 \tag{7}$$

$$\langle\langle 0|B_t^\dagger B_s|0\rangle\rangle = 0 \qquad \langle\langle 0|B_t B_s^\dagger|0\rangle\rangle = \min(t, s) \tag{8}$$

where we used (2) and (1). Since the moments (7) and (8) indicate that the operators B_t and $B_t^\dagger$ on the Fock space Γ^0 can be interpreted as the Wiener process for a quantum system, we call the operators the quantum Wiener processes‡.

2.3. Product rules

Let us introduce the *exponential vectors* $|e(f)\rangle\rangle, \langle\langle e(f)| \in \Gamma^0$ by

$$|e(f)\rangle\rangle = \exp\left[\int_0^\infty \mathrm{d}t\, f(t) b^\dagger(t)\right]|0\rangle\rangle \qquad \langle\langle e(f)| = \langle\langle 0| \exp\left[\int_0^\infty \mathrm{d}t\, f^*(t) b(t)\right] \tag{9}$$

where f is an element of the set L^2 of square integrable functions satisfying $\int_0^\infty \mathrm{d}t\, |f(t)|^2 < \infty$. Since the sets $\{|e(f)\rangle\rangle | f \in L^2\}$ and $\{\langle\langle e(f)|| f \in L^2\}$ of all exponential vectors are *linearly independent* and *total* in the Fock space Γ^0 [10], any operator on the Fock space is characterized by the action on the exponential vectors [5]. The annihilation and creation operators $b(t)$ and $b^\dagger(t)$ are characterized by the relations

$$b(t)|e(f)\rangle\rangle = f(t)|e(f)\rangle\rangle \qquad \langle\langle e(f)|b^\dagger(t) = \langle\langle e(f)|f^*(t) \tag{10}$$

respectively.

† Since annihilation and creation operators $b(t)$ and $b^\dagger(t)$ satisfy bosonic canonical commutation relations (1), the vector space Γ^0 is also called the *boson Fock space* or the *symmetric Fock space* [10].

‡ The processes B_t and $B_t^\dagger$ are also called annihilation and creation processes, respectively [9, 10].

With the help of the properties (10), the increments $\mathrm{d}B_t = B_{t+\mathrm{d}t} - B_t$, $\mathrm{d}B_t^\dagger = B_{t+\mathrm{d}t}^\dagger - B_t^\dagger$ of the quantum Wiener processes B_t and $B_t^\dagger$ defined by (6) are characterized by the following relations:

$$\langle\langle e(f)|\mathrm{d}B_t|e(f')\rangle\rangle = f'(t)\,\mathrm{d}t\,\langle\langle e(f)|e(f')\rangle\rangle \tag{11}$$

$$\langle\langle e(f)|\mathrm{d}B_t^\dagger|e(f')\rangle\rangle = f^*(t)\,\mathrm{d}t\,\langle\langle e(f)|e(f')\rangle\rangle. \tag{12}$$

The products of the increments $\mathrm{d}B_t$, $\mathrm{d}B_t^\dagger$ and $\mathrm{d}t$ are characterized by the following relations†:

$$\langle\langle e(f)|\mathrm{d}B_t\,\mathrm{d}B_t|e(f')\rangle\rangle = O(\mathrm{d}t^2) \tag{13}$$

$$\langle\langle e(f)|\mathrm{d}B_t\,\mathrm{d}B_t^\dagger|e(f')\rangle\rangle = \mathrm{d}t\,\langle\langle e(f)|e(f')\rangle\rangle + O(\mathrm{d}t^2) \qquad \text{etc.} \tag{14}$$

Taking into account the terms of $O(\mathrm{d}t)$ in L^2-space and neglecting the terms of $o(\mathrm{d}t)$, we have, from the matrix elements (11)–(14), the following product rules [5]:

	$\mathrm{d}B_t$	$\mathrm{d}B_t^\dagger$	$\mathrm{d}t$
$\mathrm{d}B_t$	0	$\mathrm{d}t$	0
$\mathrm{d}B_t^\dagger$	0	0	0
$\mathrm{d}t$	0	0	0

(15)

2.4. *Thermal space*

We introduce the *tilde* operators $(\tilde{b}(t), \tilde{b}^\dagger(t))$ on the space $\tilde{\Gamma}^0$ which is a *tilde conjugate space* of Γ^0 associated with $(b(t), b^\dagger(t))$. Here, the *tilde conjugation* $\tilde{}$ is defined by the following rules.

(1) For arbitrary operators A_1, A_2 and A, complex c-numbers c_1 and c_2, we have

$$(A_1 A_2)\tilde{} = \tilde{A}_1 \tilde{A}_2 \tag{16}$$

$$(c_1 A_1 + c_2 A_2)\tilde{} = c_1^* \tilde{A}_1 + c_2^* \tilde{A}_2 \tag{17}$$

$$(\tilde{A})\tilde{} = A \tag{18}$$

$$(A^\dagger)\tilde{} = \tilde{A}^\dagger. \tag{19}$$

(2) The tilde and non-tilde operators in the Schrödinger representation are mutually commutative:

$$[A, \tilde{B}] = 0. \tag{20}$$

Let the vacuums in $\tilde{\Gamma}^0$ be denoted by $|\tilde{0}\rangle\rangle$ and $\langle\langle\tilde{0}|$ which are defined by

$$\tilde{b}(t)|\tilde{0}\rangle\rangle = 0 \qquad \langle\langle\tilde{0}|\tilde{b}^\dagger(t) = 0. \tag{21}$$

The tilde conjugate space $\tilde{\Gamma}^0$ is the Fock space spanned by the basic vectors which are introduced by cyclic operations of $\tilde{b}^\dagger(t)$ on the vacuum $|\tilde{0}\rangle\rangle$ and $\tilde{b}(t)$ on the vacuum $\langle\langle\tilde{0}|$.

Now, we consider a tensor product space

$$\Gamma = \Gamma^0 \otimes \tilde{\Gamma}^0. \tag{22}$$

† $O(x)$ indicates that

$$\lim_{x\to 0} \frac{O(x)}{x} = \alpha \neq 0$$

while $o(x)$ indicates that

$$\lim_{x\to 0} \frac{o(x)}{x} = 0.$$

The vacuum states $|0)$ and $(0|$ of Γ are defined by

$$b(t)|0) = \tilde{b}(t)|0) = 0 \qquad (0|b^\dagger(t) = (0|\tilde{b}^\dagger(t) = 0 \tag{23}$$

where we have used the notational conventions such as

$$b(t) \otimes \tilde{I} \Rightarrow b(t) \qquad b^\dagger(t) \otimes \tilde{I} \Rightarrow b^\dagger(t) \tag{24}$$

$$I \otimes \tilde{b}(t) \Rightarrow \tilde{b}(t) \qquad I \otimes \tilde{b}^\dagger(t) \Rightarrow \tilde{b}^\dagger(t) \tag{25}$$

where I and $\tilde{I}$ stand for identity operators on Γ^0 and $\tilde{\Gamma}^0$, respectively. In the following, we will use the conventions. The vacuums $|0)$ and $(0|$ can be written as

$$|0) = |0\rangle\rangle \otimes |\tilde{0}\rangle\rangle \qquad (0| = \langle\langle 0| \otimes \langle\langle \tilde{0}|. \tag{26}$$

Γ is the Fock space spanned by the basic vectors which are introduced by cyclic operations of $(b^\dagger(t), \tilde{b}^\dagger(t))$ on the vacuum $|0)$ and $(b(t), \tilde{b}(t))$ on the vacuum $(0|$. The annihilation and creation operators $b(t)$, $b^\dagger(t)$, $\tilde{b}(t)$ and $\tilde{b}^\dagger(t)$ on Γ satisfy the canonical commutation relations

$$[b(t), b^\dagger(s)] = [\tilde{b}(t), \tilde{b}^\dagger(s)] = \delta(t-s) \tag{27}$$

$$[b(t),\ b(s)] = [\tilde{b}(t), \tilde{b}(s)] = [b(t), \tilde{b}(s)] = [b(t), \tilde{b}^\dagger(s)] = 0. \tag{28}$$

The thermal degree of freedom can be introduced by *Bogoliubov transformation* in Γ. First, we require that the expectation value of $b^\dagger(t)b(s)$ should be

$$\langle b^\dagger(t)b(s)\rangle = \bar{n}\delta(t-s) \tag{29}$$

with a real positive number $\bar{n}$, where $\langle\cdots\rangle$ indicates the expectation with respect to *thermal ket-vacuum* $|\rangle$ and *thermal bra-vacuum* $\langle|$. We find that in order to ensure equation (29), it is sufficient to impose the *thermal state conditions* on the states $|\rangle$ and $\langle|$:

$$b(t)|\rangle = \frac{\bar{n}}{1+\bar{n}}\tilde{b}^\dagger(t)|\rangle \qquad \langle|b^\dagger(t) = \langle|\tilde{b}(t). \tag{30}$$

In fact, using the conditions (30), we have

$$\langle b^\dagger(t)b(s)\rangle = \frac{\bar{n}}{1+\bar{n}}\{\langle b^\dagger(t)b(s)\rangle + \delta(t-s)\} \tag{31}$$

which leads to (29).

We introduce annihilation operators $(c(t), \tilde{c}(t))$ and creation operators $(c^\natural(t), \tilde{c}^\natural(t))$ for the thermal ket-vacuum $|\rangle$ satisfying

$$c(t)|\rangle = \tilde{c}(t)|\rangle = 0 \qquad \langle|c^\natural(t) = \langle|\tilde{c}^\natural(t) = 0 \tag{32}$$

and the canonical commutation relations

$$[c(t), c^\natural(s)] = [\tilde{c}(t), \tilde{c}^\natural(s)] = \delta(t-s) \tag{33}$$

$$[c(t), c(s)] = [\tilde{c}(t), \tilde{c}(s)] = [c(t), \tilde{c}(s)] = [c(t), \tilde{c}^\natural(s)] = 0 \tag{34}$$

$$[c^\natural(t), c^\natural(s)] = [\tilde{c}^\natural(t), \tilde{c}^\natural(s)] = [c^\natural(t), \tilde{c}(s)] = [c^\natural(t), \tilde{c}^\natural(s)] = 0. \tag{35}$$

Recalling the thermal state conditions (30), we see that such operators $(c(t), c^\natural(t))$ and their tilde conjugates are related to $(b(t), b^\dagger(t))$ and their tilde conjugates through the *Bogoliubov transformation* [19]

$$\begin{pmatrix} c(t) \\ \tilde{c}^\natural(t) \end{pmatrix} = \begin{pmatrix} 1+\bar{n} & -\bar{n} \\ -1 & 1 \end{pmatrix} \begin{pmatrix} b(t) \\ \tilde{b}^\dagger(t) \end{pmatrix}. \tag{36}$$

The Bogoliubov transformation is the canonical one such that the canonical commutation relations do not change under this transformation.

Let Γ^{β} denote the boson Fock space spanned by the basic ket- and bra-vectors introduced by cyclic operations of $(c^{\natural}(t), \tilde{c}^{\natural}(t))$ on the thermal ket-vacuum $|\rangle$ and of $(c(t), \tilde{c}(t))$ on the thermal bra-vacuum $\langle|$.

The Bogoliubov transformation (36) is generated by

$$U_{\mathrm{B}} = \exp\left[-\bar{n} \int_0^{\infty} \mathrm{d}t\, b^{\dagger}(t)\tilde{b}^{\dagger}(t) \right] \exp\left[\int_0^{\infty} \mathrm{d}t\, b(t)\tilde{b}(t) \right] \tag{37}$$

$$U_{\mathrm{B}}^{-1} = \exp\left[-\int_0^{\infty} \mathrm{d}t\, b(t)\tilde{b}(t) \right] \exp\left[\bar{n} \int_0^{\infty} \mathrm{d}t\, b^{\dagger}(t)\tilde{b}^{\dagger}(t) \right] \tag{38}$$

as

$$c(t) = U_{\mathrm{B}}^{-1} b(t) U_{\mathrm{B}} \qquad \tilde{c}^{\natural}(t) = U_{\mathrm{B}}^{-1} \tilde{b}^{\dagger}(t) U_{\mathrm{B}}. \tag{39}$$

Equations (39) together with the properties (23) and (32) give formally the relations between the thermal vacuums in Γ^{β} and the vacuums in Γ as follows:

$$|\rangle = U_{\mathrm{B}}^{-1}|0) \qquad \langle| = (0|U_{\mathrm{B}}. \tag{40}$$

Using the well known formula of the Lie algebra of the SU(1, 1) group, we can rewrite U_{B}^{-1} as a normal ordered product

$$\begin{aligned} U_{\mathrm{B}}^{-1} = \exp&\left[\frac{\bar{n}}{1+\bar{n}} \int_0^{\infty} \mathrm{d}t\, b^{\dagger}(t)\tilde{b}^{\dagger}(t) \right] \\ &\times \exp\left[-\ln(1+\bar{n}) \int_0^{\infty} \mathrm{d}t \{ b^{\dagger}(t)b(t) + \tilde{b}^{\dagger}(t)\tilde{b}(t) + \delta(0) \} \right] \\ &\times \exp\left[-\frac{1}{1+\bar{n}} \int_0^{\infty} \mathrm{d}t\, b(t)\tilde{b}(t) \right]. \end{aligned} \tag{41}$$

Here, $\delta(0)$ is the delta function $\delta(t)$ with $t = 0$. Equations (40) and (41) together with the property (23) give

$$|\rangle = \exp\left[-\delta(0)\ln(1+\bar{n}) \int_0^{\infty} \mathrm{d}t \right] \exp\left[\frac{\bar{n}}{1+\bar{n}} \int_0^{\infty} \mathrm{d}t\, b^{\dagger}(t)\tilde{b}^{\dagger}(t) \right] |0). \tag{42}$$

Since $\delta(0) = \infty$, equation (42) shows that any expansion coefficients of the thermal ket-vacuum $|\rangle$ expanded by the complete orthonormal basis in Γ vanish. Namely, in the thermal ket-vacuum $|\rangle$, an infinite number of the *thermal pairs*, which are particle pairs created by the operators $b^{\dagger}(t)\tilde{b}^{\dagger}(t)$, are condensed and the Fock space Γ^{β} is *inequivalent* to the Fock space Γ in the sense that any vector in Γ^{β} cannot be written as a superposition of vectors in Γ and *vice versa*.

On the other hand, equation (40) together with the expression (37) of U_{B} gives

$$\begin{aligned} \langle| &= (0| \exp\left[-\bar{n} \int_0^{\infty} \mathrm{d}t\, b^{\dagger}(t)\tilde{b}^{\dagger}(t) \right] \exp\left[\int_0^{\infty} \mathrm{d}t\, b(t)\tilde{b}(t) \right] \\ &= (0| \exp\left[\int_0^{\infty} \mathrm{d}t\, b(t)\tilde{b}(t) \right] \end{aligned} \tag{43}$$

where we have used property (23). We see that equation (43) is consistent with the thermal state condition (30) of the bra-vacuum. In fact, using equation (43) and property (23), we can prove the thermal state condition (30).

2.5. Quantum Wiener processes at finite temperatures

Quantum Wiener processes at finite temperatures are defined by the operators

$$B_t = \int_0^t \mathrm{d}s\, b(s) \qquad B_t^\dagger = \int_0^t \mathrm{d}s\, b^\dagger(s) \tag{44}$$

and their tilde conjugates represented in the Fock space Γ^β. The explicit representations of the processes B_t, $B_t^\dagger$, $\tilde{B}_t$ and $\tilde{B}_t^\dagger$ in Γ^β are given in terms of the Bogoliubov transformation (36) by

$$B_t = \int_0^t \mathrm{d}s[c(s) + \bar{n}\tilde{c}^{\natural}(s)] = C_t + \bar{n}\tilde{C}_t^{\natural} \tag{45}$$

$$B_t^\dagger = \int_0^t \mathrm{d}s[\tilde{c}(s) + (1+\bar{n})c^{\natural}(s)] = \tilde{C}_t + (1+\bar{n})C_t^{\natural} \tag{46}$$

and their tilde conjugates, where C_t, $C_t^{\natural}$, $\tilde{C}_t$ and $\tilde{C}_t^{\natural}$ are the annihilation and creation processes in Γ^β defined by

$$C_t = \int_0^t \mathrm{d}s\, c(s) \qquad C_t^{\natural} = \int_0^t \mathrm{d}s\, c^{\natural}(s) \tag{47}$$

and their tilde conjugates.

Any operator in the Fock space Γ^β can be characterized by the exponential vectors $|e(f,g)\rangle$, $\langle e(f,g)|$ in Γ^β with $f, g \in L^2$ defined by

$$|e(f,g)\rangle = \exp\left[\int_0^\infty \mathrm{d}t\{f(t)c^{\natural}(t) + g^*(s)\tilde{c}^{\natural}(s)\}\right]|\rangle \tag{48}$$

$$\langle e(f,g)| = \langle|\exp\left[\int_0^\infty \mathrm{d}t\{f^*(t)c(t) + g(s)\tilde{c}(s)\}\right] \tag{49}$$

which satisfy the following relations:

$$c(t)|e(f,g)\rangle = f(t)|e(f,g)\rangle \qquad \langle e(f,g)|c^{\natural}(t) = \langle e(f,g)|f^*(t) \tag{50}$$

$$\tilde{c}(t)|e(f,g)\rangle = g^*(t)|e(f,g)\rangle \qquad \langle e(f,g)|\tilde{c}^{\natural}(t) = \langle e(f,g)|g(t). \tag{51}$$

As in the case of the construction of annihilation and creation processes $(B_t, B_t^\dagger)$ on Γ^0, an evaluation of matrix elements of the products among the increments $\mathrm{d}C_t$, $\mathrm{d}C_t^{\natural}$, $\mathrm{d}\tilde{C}_t$, $\mathrm{d}\tilde{C}_t^{\natural}$ and $\mathrm{d}t$, with the help of properties (50) and (51), gives the following product rules:

	$\mathrm{d}C_t$	$\mathrm{d}C_t^{\natural}$	$\mathrm{d}\tilde{C}_t$	$\mathrm{d}\tilde{C}_t^{\natural}$	$\mathrm{d}t$
$\mathrm{d}C_t$	0	$\mathrm{d}t$	0	0	0
$\mathrm{d}C_t^{\natural}$	0	0	0	0	0
$\mathrm{d}\tilde{C}_t$	0	0	0	$\mathrm{d}t$	0
$\mathrm{d}\tilde{C}_t^{\natural}$	0	0	0	0	0
$\mathrm{d}t$	0	0	0	0	0

(52)

By making use of the expressions (45), (46) and their tilde conjugates, and the product rules (52), we can evaluate the products $\mathrm{d}B_t\,\mathrm{d}B_t^\dagger$ as

$$\begin{aligned}\mathrm{d}B_t\,\mathrm{d}B_t^\dagger &= [\mathrm{d}C_t + \bar{n}\,\mathrm{d}\tilde{C}_t^{\natural}][\mathrm{d}\tilde{C}_t + (\bar{n}+1)\,\mathrm{d}C_t^{\natural}] \\ &= \mathrm{d}C_t\,\mathrm{d}\tilde{C}_t + (\bar{n}+1)\,\mathrm{d}C_t\,\mathrm{d}C_t^{\natural} + \bar{n}\,\mathrm{d}\tilde{C}_t^{\natural}\,\mathrm{d}\tilde{C}_t + \bar{n}(\bar{n}+1)\,\mathrm{d}\tilde{C}_t^{\natural}\,\mathrm{d}C_t^{\natural} \\ &= (\bar{n}+1)\,\mathrm{d}t.\end{aligned} \tag{53}$$

Similarly, we can evaluate the other products of the increments of $\mathrm{d}B_t$, $\mathrm{d}B_t^\dagger$, $\mathrm{d}\tilde{B}_t$, $\mathrm{d}\tilde{B}_t^\dagger$ and $\mathrm{d}t$ and obtain the product rules summarized as follows.

	$\mathrm{d}B_t$	$\mathrm{d}B_t^\dagger$	$\mathrm{d}\tilde{B}_t$	$\mathrm{d}\tilde{B}_t^\dagger$	$\mathrm{d}t$
$\mathrm{d}B_t$	0	$(1+\bar{n})\,\mathrm{d}t$	$\bar{n}\,\mathrm{d}t$	0	0
$\mathrm{d}B_t^\dagger$	$\bar{n}\,\mathrm{d}t$	0	0	$(1+\bar{n})\,\mathrm{d}t$	0
$\mathrm{d}\tilde{B}_t$	$\bar{n}\,\mathrm{d}t$	0	0	$(1+\bar{n})\,\mathrm{d}t$	0
$\mathrm{d}\tilde{B}_t^\dagger$	0	$(1+\bar{n})\,\mathrm{d}t$	$\bar{n}\,\mathrm{d}t$	0	0
$\mathrm{d}t$	0	0	0	0	0

(54)

Using equations (45), (46) and their tilde conjugates, the commutation relation (33) and the properties (32) of the thermal vacuums, we obtain the moments of the increments $\mathrm{d}B_t$, $\mathrm{d}B_t^\dagger$, $\mathrm{d}\tilde{B}_t$ and $\mathrm{d}\tilde{B}_t^\dagger$ with respect to the thermal vacuums $|\rangle$ and $\langle|$ as follows:

$$\langle \mathrm{d}B_t\rangle = \langle \mathrm{d}B_t^\dagger\rangle = \langle \mathrm{d}\tilde{B}_t\rangle = \langle \mathrm{d}\tilde{B}_t^\dagger\rangle = 0 \tag{55}$$

$$\langle \mathrm{d}B_t^\dagger\,\mathrm{d}B_s\rangle = \langle \mathrm{d}\tilde{B}_t^\dagger\,\mathrm{d}\tilde{B}_s\rangle = \langle \mathrm{d}B_t\,\mathrm{d}\tilde{B}_s\rangle = \langle \mathrm{d}\tilde{B}_t\,\mathrm{d}B_s\rangle = \bar{n}\delta(t-s)\,\mathrm{d}t\,\mathrm{d}s \tag{56}$$

$$\langle \mathrm{d}B_t\,\mathrm{d}B_s^\dagger\rangle = \langle \mathrm{d}\tilde{B}_t\,\mathrm{d}\tilde{B}_s^\dagger\rangle = \langle \mathrm{d}B_t^\dagger\,\mathrm{d}\tilde{B}_s^\dagger\rangle = \langle \mathrm{d}\tilde{B}_t^\dagger\,\mathrm{d}B_s^\dagger\rangle = (1+\bar{n})\delta(t-s)\,\mathrm{d}t\,\mathrm{d}s \tag{57}$$

$$(\text{others}) = 0. \tag{58}$$

Letting $\bar{n}$ in (56) and (57) be the Planck distribution given by

$$\bar{n} = \frac{1}{\mathrm{e}^{\beta\omega}-1} \tag{59}$$

with some positive number ω and the inverse of the temperature $\beta = 1/T$, we see that the quantum Wiener processes B_t and $B_t^\dagger$ are essentially equivalent to those introduced in the problem of quantum optics [3].

3. Quantum stochastic calculus

On the basis of the quantum Wiener processes at finite temperatures, we will investigate the quantum stochastic calculus.

3.1. Adapted processes

The Fock space Γ^β is decomposed as

$$\Gamma^\beta = \Gamma_{t]}^\beta \otimes \Gamma_{(t}^\beta \tag{60}$$

in which, for $f, g \in L^2$,

$$|e(f,g)\rangle = |e(f_{t]}, g_{t]})\rangle \otimes |e(f_{(t}, g_{(t})\rangle \qquad \langle e(f,g)| = \langle e(f_{t]}, g_{t]})| \otimes \langle e(f_{(t}, g_{(t})| \tag{61}$$

where we set

$$f_{t]} = f\chi_{t]} \qquad f_{(t} = f\chi_{(t} \tag{62}$$

and

$$|\rangle = |_{t]}\rangle \otimes |_{(t}\rangle \qquad \langle| = \langle_{t]}| \otimes \langle_{(t}|. \tag{63}$$

Here, $\chi_{t]}$ and $\chi_{(t}$ are defined by

$$\chi_{t]}(s) = \theta(t-s) \qquad \chi_{(t}(s) = \theta(s-t) \qquad \text{for } t, s > 0. \tag{64}$$

Note that $\Gamma^{\beta}_{t]}$ is the boson Fock space built on the vacuums $|_{t]}\rangle$ and $\langle_{t]}|$, while $\Gamma^{\beta}_{(t]}$ is the boson Fock space built on the vacuums $|_{(t}\rangle$ and $\langle_{(t}|$. The quantum Wiener processes B_t, $B_t^{\dagger}$, $\tilde{B}_t$ and $\tilde{B}_t^{\dagger}$ are operators on the space $\Gamma^{\beta}_{t]}$.

Let us consider a space of tensor product $\mathcal{H}_S \otimes \Gamma^{\beta}$, where $\mathcal{H}_S$ indicates a certain vector space. For the sake of notational convenience, we identify the quantum Wiener processes B_t, $B_t^{\dagger}$, $\tilde{B}_t$ and $\tilde{B}_t^{\dagger}$ with the operators on $\mathcal{H}_S \otimes \Gamma^{\beta}$, i.e.

$$I_S \otimes (B_t \otimes I_{(t}) \Rightarrow B_t \qquad I_S \otimes (B_t^{\dagger} \otimes I_{(t}) \Rightarrow B_t^{\dagger} \tag{65}$$

$$I_S \otimes (\tilde{B}_t \otimes I_{(t}) \Rightarrow \tilde{B}_t \qquad I_S \otimes (\tilde{B}_t^{\dagger} \otimes I_{(t}) \Rightarrow \tilde{B}_t^{\dagger} \tag{66}$$

where I_S and $I_{(t}$ are the identity operators on $\mathcal{H}_S$ and $\Gamma^{\beta}_{(t}$, respectively.

An *adapted process* F_t is defined by

$$F_t = F_t^0 \otimes I_{(t} \tag{67}$$

where F_t^0 is an operator on $\mathcal{H}_S \otimes \Gamma^{\beta}_{t]}$. According to the notation (65) and (66), the increments $\mathrm{d}B_t$, $\mathrm{d}B_t^{\dagger}$, $\mathrm{d}\tilde{B}_t$ and $\mathrm{d}\tilde{B}_t^{\dagger}$ on $\Gamma^{\beta}_{(t,t+\mathrm{d}t]}$ are identified with the operators on $\mathcal{H}_S \otimes \Gamma^{\beta}$, i.e.

$$I_S \otimes (I_{t]} \otimes \mathrm{d}B_t \otimes I_{(t+\mathrm{d}t}) \Rightarrow \mathrm{d}B_t \qquad I_S \otimes (I_{t]} \otimes \mathrm{d}B_t^{\dagger} \otimes I_{(t+\mathrm{d}t}) \Rightarrow \mathrm{d}B_t^{\dagger} \tag{68}$$

$$I_S \otimes (I_{t]} \otimes \mathrm{d}\tilde{B}_t \otimes I_{(t+\mathrm{d}t}) \Rightarrow \mathrm{d}\tilde{B}_t \qquad I_S \otimes (I_{t]} \otimes \tilde{B}_t^{\dagger} \otimes I_{(t+\mathrm{d}t}) \Rightarrow \mathrm{d}\tilde{B}_t^{\dagger} \tag{69}$$

where $I_{t]}$ and $I_{(t+\mathrm{d}t}$ are the identity operators on $\Gamma^{\beta}_{t]}$ and $\Gamma^{\beta}_{(t+\mathrm{d}t}$, respectively. Therefore, for an adapted process F_t, we have

$$[F_t, \mathrm{d}B_t] = [F_t, \mathrm{d}B_t^{\dagger}] = [F_t, \mathrm{d}\tilde{B}_t] = [F_t, \mathrm{d}\tilde{B}_t^{\dagger}] = 0. \tag{70}$$

Note that from (65) and (66) the quantum Wiener processes B_t, $B_t^{\dagger}$, $\tilde{B}_t$ and $\tilde{B}_t^{\dagger}$ are adapted.

3.2. Quantum stochastic integrals

Let us consider a case where X_t and $\bar{B}_t^{\#}$ denote, respectively, an arbitrary adapted process on $\mathcal{H}_S \otimes \Gamma^{\beta}$ and one of the quantum Wiener processes B_t, $B_t^{\dagger}$, $\tilde{B}_t$ and $\tilde{B}_t^{\dagger}$.

Remember that quantum stochastic integrals of Ito type are defined by

$$\int_0^T X_t \,\mathrm{d}\bar{B}_t^{\#} \equiv \lim \sum_{i=0}^{I-1} X_{t_i} (\bar{B}^{\#}_{t_{i+1}} - \bar{B}^{\#}_{t_i}) \tag{71}$$

and

$$\int_0^T \mathrm{d}\bar{B}_t^{\#} \, X_t \equiv \lim \sum_{i=0}^{I-1} (\bar{B}^{\#}_{t_{i+1}} - \bar{B}^{\#}_{t_i}) X_{t_i} \tag{72}$$

while those of the Stratonovich type are defined by

$$\int_0^T X_t \circ \mathrm{d}\bar{B}_t^{\#} \equiv \lim \sum_{i=0}^{I-1} \frac{X_{t_{i+1}} + X_{t_i}}{2} (\bar{B}^{\#}_{t_{i+1}} - \bar{B}^{\#}_{t_i}) \tag{73}$$

and

$$\int_0^T \mathrm{d}\bar{B}_t^{\#} \circ X_t \equiv \lim \sum_{i=0}^{I-1} (\bar{B}^{\#}_{t_{i+1}} - \bar{B}^{\#}_{t_i}) \frac{X_{t_{i+1}} + X_{t_i}}{2}. \tag{74}$$

We have introduced the following differential notations

$$X_t\,\mathrm{d}\bar{B}_t^{\#} \equiv X_t(\bar{B}_{t+\mathrm{d}t}^{\#} - \bar{B}_t^{\#}) \tag{75}$$

$$\mathrm{d}\bar{B}_t^{\#} X_t \equiv (\bar{B}_{t+\mathrm{d}t}^{\#} - \bar{B}_t^{\#}) X_t \tag{76}$$

$$X_t \circ \mathrm{d}\bar{B}_t^{\#} \equiv \frac{X_{t+\mathrm{d}t} + X_t}{2}(\bar{B}_{t+dt}^{\#} - \bar{B}_t^{\#}) \tag{77}$$

and

$$\mathrm{d}\bar{B}_t^{\#} \circ X_t \equiv (\bar{B}_{t+\mathrm{d}t}^{\#} - \bar{B}_t^{\#})\frac{X_{t+\mathrm{d}t} + X_t}{2}. \tag{78}$$

Here, $t_i = i\,\Delta t$ and the symbol lim indicates taking the limit

$$\Delta t \longrightarrow +0 \qquad I \longrightarrow +\infty \tag{79}$$

keeping $T = I\,\Delta t$ fixed. We call (75) and (76) the products of Ito type, whereas we refer to (77) and (78) as the products of Stratonovich type. Note that quantum stochastic integrals both of Ito and of Stratonovich types are adapted processes.

Property (70) gives us

$$[X_t, \mathrm{d}\bar{B}_t^{\#}] = 0 \tag{80}$$

in the stochastic calculus of Ito type. Therefore, we have

$$\int_0^T X_t\,\mathrm{d}\bar{B}_t^{\#} = \int_0^T \mathrm{d}\bar{B}_t^{\#}\,X_t. \tag{81}$$

In addition, using property (63) of the thermal vacuums and the property

$$\langle|\bar{B}_t^{\#}|\rangle = 0 \tag{82}$$

of the quantum Wiener processes, we see that

$$\langle|\mathrm{d}\bar{B}_t^{\#} X_t|\rangle = \langle|X_t\,\mathrm{d}\bar{B}_t^{\#}|\rangle = \langle_{t]}|X_t|_{t]}\rangle\langle_{(t}|\mathrm{d}\bar{B}_t^{\#}|_{(t}\rangle = 0. \tag{83}$$

This indicates that there is no correlation between X_t and $\mathrm{d}\bar{B}_t^{\#}$.

It should be pointed out that the increment $\mathrm{d}\bar{B}_t^{\#}$ does not commute with $X_{t+\mathrm{d}t}$, whereas it does with X_t. Therefore, in the stochastic calculus of Stratonovich type, the commutation relation of X_t and $\mathrm{d}\bar{B}_t^{\#}$ defined by

$$[X_t \overset{\circ}{,} \mathrm{d}\bar{B}_t^{\#}] \equiv X_t \circ \mathrm{d}\bar{B}_t^{\#} - \mathrm{d}\bar{B}_t^{\#} \circ X_t \tag{84}$$

is not equal to zero, i.e.

$$[X_t \overset{\circ}{,} \mathrm{d}\bar{B}_t^{\#}] \neq 0. \tag{85}$$

Therefore, in this case we have

$$\int_0^T X_t \circ \mathrm{d}\bar{B}_t^{\#} \neq \int_0^T \mathrm{d}\bar{B}_t^{\#} \circ X_t. \tag{86}$$

Furthermore, in contrast with the case of Ito type,

$$\langle|X_t \circ \mathrm{d}\bar{B}_t^{\#}|\rangle \neq 0 \qquad \langle|\mathrm{d}\bar{B}_t^{\#} \circ X_t|\rangle \neq 0. \tag{87}$$

Substitution $X_{t+\mathrm{d}t} = X_t + \mathrm{d}X_t$ into (77) and (78) gives us the relations between the products of Ito and Stratonovich types in the form

$$X_t \circ \mathrm{d}\bar{B}_t^{\#} = X_t\,\mathrm{d}\bar{B}_t^{\#} + \tfrac{1}{2}\,\mathrm{d}X_t\,\mathrm{d}\bar{B}_t^{\#} \tag{88}$$

and

$$\mathrm{d}\bar{B}_t^{\#} \circ X_t = \mathrm{d}\bar{B}_t^{\#} X_t + \tfrac{1}{2}\,\mathrm{d}\bar{B}_t^{\#}\,\mathrm{d}X_t. \tag{89}$$

3.3. Quantum Ito's formula

Let us consider an operator N_t representing an adapted process $[N_t, \mathrm{d}\bar{B}_t^{\#}] = 0$;

$$N_T = \int_0^T (F_t\, \mathrm{d}B_t + G_t\, \mathrm{d}B_t^{\dagger} + J_t\, \mathrm{d}\tilde{B}_t + K_t\, \mathrm{d}\tilde{B}_t^{\dagger} + H_t\, \mathrm{d}t) \tag{90}$$

where F_t, G_t, H_t, J_t and K_t are adapted processes. Its differential notation is given by

$$\mathrm{d}N_t = F_t\, \mathrm{d}B_t + G_t\, \mathrm{d}B_t^{\dagger} + J_t\, \mathrm{d}\tilde{B}_t + K_t\, \mathrm{d}\tilde{B}_t^{\dagger} + H_t\, \mathrm{d}t. \tag{91}$$

It should be noted that the increment $\mathrm{d}N_t$ does not commute with an arbitrary adapted process X_t in general, i.e.

$$[X_t, \mathrm{d}N_t] \neq 0 \tag{92}$$

because $\mathrm{d}N_t$ includes not only the increments $\mathrm{d}B_t$, $\mathrm{d}B_t^{\dagger}$, $\mathrm{d}\tilde{B}_t$ and $\mathrm{d}\tilde{B}_t^{\dagger}$ but also the adapted processes F_t, G_t, J_t, K_t and H_t. Furthermore, for the adapted process X_t and the increment $\mathrm{d}N_t$, the property, such as (83), does not hold because of the term $H_t\, \mathrm{d}t$†, i.e.

$$\langle | X_t\, \mathrm{d}N_t | \rangle = \langle | X_t H_t | \rangle\, \mathrm{d}t \neq 0 \qquad \langle | \mathrm{d}N_t X_t | \rangle = \langle | H_t X_t | \rangle\, \mathrm{d}t \neq 0. \tag{93}$$

Let N_t' be another stochastic integral defined by

$$N_T' = \int_0^T (F_t'\, \mathrm{d}B_t + G_t'\, \mathrm{d}B_t^{\dagger} + J_t'\, \mathrm{d}\tilde{B}_t + K_t'\, \mathrm{d}\tilde{B}_t^{\dagger} + H_t'\, \mathrm{d}t) \tag{94}$$

with adapted processes F_t', G_t', H_t', J_t' and K_t', of which the differential notation is given by

$$\mathrm{d}N_t' = F_t'\, \mathrm{d}B_t + G_t'\, \mathrm{d}B_t^{\dagger} + J_t'\, \mathrm{d}\tilde{B}_t + K_t'\, \mathrm{d}\tilde{B}_t^{\dagger} + H_t'\, \mathrm{d}t. \tag{95}$$

The product $N_{t+\mathrm{d}t} N_{t+\mathrm{d}t}'$ is calculated as

$$N_{t+\mathrm{d}t} N_{t+\mathrm{d}t}' = (N_t + \mathrm{d}N_t)(N_t' + \mathrm{d}N_t') = N_t N_t' + N_t\, \mathrm{d}N_t' + \mathrm{d}N_t N_t' + \mathrm{d}N_t\, \mathrm{d}N_t' \tag{96}$$

where $\mathrm{d}N_t$ and $\mathrm{d}N_t'$ are given by (91) and (95), respectively. In contrast with the ordinary calculus, the last term $\mathrm{d}N_t\, \mathrm{d}N_t'$ of (96) is not the order of $o(\mathrm{d}t)$. In fact, using property (70) and the product rules (54), we obtain

$$\mathrm{d}N_t\, \mathrm{d}N_t' = [(1+\bar{n})(F_t G_t' + G_t K_t' + J_t K_t' + K_t G_t') + \bar{n}(F_t J_t' + G_t F_t' + J_t F_t' + K_t J_t')]\, \mathrm{d}t. \tag{97}$$

With the help of expressions (91) and (95) together with (97), we find that (96) gives

$$\begin{aligned}
\mathrm{d}(N_t N_t') = {} & (N_t F_t' + F_t N_t')\, \mathrm{d}B_t + (N_t G_t' + G_t N_t')\, \mathrm{d}B_t^{\dagger} \\
& + (N_t J_t' + J_t N_t')\, \mathrm{d}\tilde{B}_t + (N_t K_t' + K_t N_t')\, \mathrm{d}\tilde{B}_t^{\dagger} \\
& + [N_t H_t' + H_t N_t' + (1+\bar{n})(F_t G_t' + G_t K_t' + J_t K_t' + K_t G_t') \\
& + \bar{n}(F_t J_t' + G_t F_t' + J_t F_t' + K_t J_t')]\, \mathrm{d}t.
\end{aligned} \tag{98}$$

As we saw in deriving (98), *quantum Ito's formula*

$$\mathrm{d}(N_t N_t') = \mathrm{d}N_t \cdot N_t' + N_t \cdot \mathrm{d}N_t' + \mathrm{d}N_t\, \mathrm{d}N_t' \tag{99}$$

holds for stochastic integrals N_t and N_t' defined by (90) and (94), respectively [5, 9, 10].

Making use of relations (88) and (89) between the Ito and the Stratonovich products, we have

$$N_t \circ \mathrm{d}N_t' = N_t\, \mathrm{d}N_t' + \tfrac{1}{2}\, \mathrm{d}N_t\, \mathrm{d}N_t' \tag{100}$$

† When $H_t = 0$, $\mathrm{d}N_t$ satisfies $\langle\langle 0 | X_t\, \mathrm{d}N_t | 0 \rangle\rangle = \langle\langle 0 | \mathrm{d}N_t X_t | 0 \rangle\rangle = 0$, although $\mathrm{d}N_t$ still does not commute with X_t, i.e. $[X_t, \mathrm{d}N_t] \neq 0$. When $H_t = 0$, N_t is called the *martingale* [7, 33].

and

$$\mathrm{d}N_t \circ N_t' = \mathrm{d}N_t N_t' + \tfrac{1}{2}\,\mathrm{d}N_t\,\mathrm{d}N_t'. \tag{101}$$

Therefore, we find that quantum Ito's formula (99) is expressed in terms of the Stratonovich products as

$$\mathrm{d}(N_t N_t') = \mathrm{d}N_t \circ N_t' + N_t \circ \mathrm{d}N_t' \tag{102}$$

which is identical to the well known formula of ordinary differential calculus.

4. Stochastic Schrödinger equation

In this section, we review the stochastic Schrödinger equation investigated by Hudson and Lindsay [7].

4.1. Ito type

We consider a boson system which is described by the operators a and $a^\dagger$ on a Hilbert space $\mathcal{H}_S^0$ satisfying the commutation relations

$$[a, a^\dagger] = 1 \qquad [a, a] = 0 \tag{103}$$

and which interacts with a reservoir at finite temperatures. Let us suppose that the effect of the reservoir on the system is taken into account by the random force operators represented by the quantum Wiener processes at finite temperatures constructed on the Fock space Γ^β. We sometimes call the boson system the relevant system and the reservoir system the irrelevant system.

The state of the system is described by the state vector $|\psi_f(t)\rangle\rangle$ in the space $\mathcal{H}_S^0 \otimes \Gamma^\beta$. The state vector $|\psi_f(t)\rangle\rangle$ is assumed to evolve in time according to the Schrödinger equation

$$\mathrm{d}|\psi_f(t)\rangle\rangle = -\mathrm{i}\mathcal{H}_{f,t}\,\mathrm{d}t|\psi_f(t)\rangle\rangle \tag{104}$$

with an infinitesimal time-evolution generator $\mathcal{H}_{f,t}\,\mathrm{d}t$ including random force operators. We call equation (104) the *stochastic Schrödinger equation*.

The formal solution of (104) is written by

$$|\psi_f(t)\rangle\rangle = V_f(t)|\psi_f(0)\rangle\rangle \tag{105}$$

where $V_f(t)$ is the stochastic time-evolution generator satisfying the equation

$$\mathrm{d}V_f(t) = -\mathrm{i}\mathcal{H}_{f,t}\,\mathrm{d}t\,V_f(t) \tag{106}$$

with $V_f(0) = 1$. Note that the bra-vector $\langle\langle\psi_f(t)|$ is defined by

$$\langle\langle\psi_f(t)| = \langle\langle\psi_f(0)|V_f^{-1}(t) \tag{107}$$

where $V_f^{-1}(t)$ is the inverse of $V_f(t)$.

For a bi-linear and phase invariant boson system with the interaction

$$\mathrm{i}\sqrt{2\kappa}(a^\dagger\,\mathrm{d}B_t - a\,\mathrm{d}B_t^\dagger)$$

$\mathcal{H}_{f,t}\,\mathrm{d}t$ has the form

$$\mathcal{H}_{f,t}\,\mathrm{d}t = Z\,\mathrm{d}t + \mathrm{i}\sqrt{2\kappa}(a^\dagger\,\mathrm{d}B_t - a\,\mathrm{d}B_t^\dagger) \tag{108}$$

with the relevant system operator $Z \in \mathcal{H}_S^0$. $\mathrm{d}B_t$ and $\mathrm{d}B_t^\dagger$ are the increments of the quantum Wiener processes at finite temperatures and κ is a positive c-number. Note that we adopt the same notation for B_t, $B_t^\dagger$ and their tilde conjugates $\tilde{B}_t$, $\tilde{B}_t^\dagger$ as (65) and (66). Furthermore, we use the following notation for the relevant system operators

$$Z \otimes I_R \Rightarrow Z \qquad a \otimes I_R \Rightarrow a \qquad \text{etc.} \tag{109}$$

Note that since equation (106) with the infinitesimal time-evolution generator (108) is the quantum stochastic differential equation of Ito type, the time-evolution generator $V_f(t)$ is the quantum stochastic integral of Ito type which is an adapted process.

We require that the time-evolution generator $V_f(t)$ should be unitary, i.e.

$$V_f^\dagger(t)V_f(t) = V_f(t)V_f^\dagger(t) = 1. \tag{110}$$

Therefore, we have the algebraic identities

$$\mathrm{d}[V_f^\dagger(t)V_f(t)] = \mathrm{d}V_f^\dagger(t)\cdot V_f(t) + V_f^\dagger(t)\cdot \mathrm{d}V_f(t) + \mathrm{d}V_f^\dagger(t)\,\mathrm{d}V_f(t) = 0 \tag{111}$$

and

$$\mathrm{d}[V_f(t)V_f^\dagger(t)] = \mathrm{d}V_f(t)\cdot V_f^\dagger(t) + V_f(t)\cdot \mathrm{d}V_f^\dagger(t) + \mathrm{d}V_f(t)\,\mathrm{d}V_f^\dagger(t) = 0 \tag{112}$$

where we have made use of the calculus rule of Ito type (quantum Ito's formula). The identities (111) and (112) with equation (106) and its Hermitian conjugate give the following relation,

$$\mathrm{i}(Z^\dagger - Z) + 2\kappa\left[(\bar{n}+1)a^\dagger a + \bar{n}aa^\dagger\right] = 0 \tag{113}$$

where use has been made of the product rules (54). Thus, we obtain

$$\mathcal{H}_{f,t}\,\mathrm{d}t = H_S\,\mathrm{d}t - \mathrm{i}\kappa[(1+\bar{n})a^\dagger a + \bar{n}aa^\dagger]\,\mathrm{d}t + \mathrm{i}\sqrt{2\kappa}(a^\dagger\,\mathrm{d}B_t - a\,\mathrm{d}B_t^\dagger) \tag{114}$$

where we have put $(Z + Z^\dagger)/2 = H_S$. Note that H_S is Hermitian. In the following, we will put $\bar{n}$ to the Planck distribution function (59).

Applying equation (106) of $V_f(t)$ to the state vector $|\psi_f(0)\rangle\rangle$, we have the stochastic Schrödinger equation of Ito type

$$\mathrm{d}|\psi_f(t)\rangle\rangle = -\mathrm{i}\mathcal{H}_{f,t}\,\mathrm{d}t|\psi_f(t)\rangle\rangle \tag{115}$$

with the infinitesimal time-evolution generator (114).

4.2. *Stratonovich type*

Using the relation (89) between the Ito and the Stratonovich products, we transform the stochastic differential equation (106) of Ito type into that of Stratonovich type as

$$\begin{aligned}
\mathrm{d}V_f(t) &= -\mathrm{i}\mathcal{H}_{f,t}\,\mathrm{d}t\,V_f(t)\\
&= -\mathrm{i}\{\mathcal{H}_{f,t}\,\mathrm{d}t\circ V_f(t) - \tfrac{1}{2}\mathcal{H}_{f,t}\,\mathrm{d}t\,\mathrm{d}V_f(t)\}\\
&\equiv -\mathrm{i}H_{f,t}\,\mathrm{d}t\circ V_f(t)
\end{aligned} \tag{116}$$

where we have substituted (106) into the right-hand side of the second equality. Here, we have defined the infinitesimal time-evolution generator $H_{f,t}$ of Stratonovich type by

$$H_{f,t}\,\mathrm{d}t \equiv \mathcal{H}_{f,t}\,\mathrm{d}t + \mathrm{i}\tfrac{1}{2}\mathcal{H}_{f,t}\,\mathrm{d}t\mathcal{H}_{f,t}\,\mathrm{d}t. \tag{117}$$

With the help of the product rules (54), we obtain the Hermitian stochastic infinitesimal time-evolution generator $H_{f,t}\,\mathrm{d}t$ as

$$H_{f,t}\,\mathrm{d}t = H_S\,\mathrm{d}t + \mathrm{i}\sqrt{2\kappa}(a^\dagger\,\mathrm{d}B_t - a\,\mathrm{d}B_t^\dagger). \tag{118}$$

The Hermiticity of $H_{f,t}\,\mathrm{d}t$ guarantees the unitarity of $V_f(t)$.

Applying equation (116) of $V_f(t)$ to the state vector $|\psi_f(0)\rangle\rangle$, we obtain the stochastic Schrödinger equation of Stratonovich type

$$\mathrm{d}|\psi_f(t)\rangle\rangle = -\mathrm{i}H_{f,t}\,\mathrm{d}t \circ |\psi_f(t)\rangle\rangle \tag{119}$$

with the infinitesimal time-evolution generator (118).

5. Stochastic time-evolution in thermal space

On the basis of the stochastic Schrödinger equation, investigated in the previous section, we will construct a stochastic Liouville equation in thermal space and obtain the explicit form of the time-evolution generator satisfying the stochastic Liouville equation within the framework of NETFD. Using the time-evolution generator, we will construct a unified canonical operator formalism of quantum stochastic differential equations.

5.1. Thermal vacuums

Let us define the density operator $\rho_f(t)$ corresponding to the state vector $|\psi_f(t)\rangle\rangle$ by

$$\rho_f(t) \equiv |\psi_f(t)\rangle\rangle\langle\langle\psi_f(t)|. \tag{120}$$

Using (105) and (107) with the unitary time-evolution generator $V_f(t)$, we see that (120) becomes

$$\rho_f(t) = V_f(t)|\psi_f(0)\rangle\rangle\langle\langle\psi_f(0)|V_f^\dagger(t) = V_f(t)\rho_f(0)V_f^\dagger(t). \tag{121}$$

The density operator $\rho_f(t)$ satisfies

$$\mathrm{tr}_{\mathrm{tot}}\rho_f(t) = 1 \tag{122}$$

where the trace operation $\mathrm{tr}_{\mathrm{tot}}$ is defined by

$$\mathrm{tr}_{\mathrm{tot}} \equiv \mathrm{tr} \otimes \mathrm{tr}_{\mathrm{R}} \tag{123}$$

with the trace operations tr of the relevant system and tr_{R} of the reservoir. The expectation value of any observable A is given by $\mathrm{tr}_{\mathrm{tot}}A\rho_f(t)$.

With the help of the principle of correspondence (see the appendix), the density operator $\rho_f(t)$ defined by (121) is expressed as a thermal ket-vacuum, i.e.

$$|0_f(t)\rangle \equiv |\rho_f(t)\rangle = \hat{V}_f(t)|0_f(0)\rangle \tag{124}$$

where we have defined the stochastic time-evolution generator by

$$\hat{V}_f(t) = V_f(t)\tilde{V}_f(t). \tag{125}$$

Note that, since $V_f(0) = 1$, we have $\hat{V}_f(0) = 1$. The vector space to which the thermal vacuum $|0_f(t)\rangle$ belongs is assumed to be $\mathcal{H}_S \otimes \Gamma^\beta$ where $\mathcal{H}_S$ is the tensor product space of relevant system $\mathcal{H}_S^0$ and its tilde conjugate space $\tilde{\mathcal{H}}_S^0$, i.e. $\mathcal{H}_S = \mathcal{H}_S^0 \otimes \tilde{\mathcal{H}}_S^0$, and Γ^β is the Fock space of the quantum Wiener processes at finite temperatures constructed in section 2. The operator $\hat{V}_f(t)$ defined by (125) is that on the space $\mathcal{H}_S \otimes \Gamma^\beta$ and turns out to be unitary from the relation

$$\hat{V}_f^\dagger(t) = V_f^\dagger(t)\tilde{V}_f^\dagger(t) = V_f^{-1}(t)\tilde{V}_f^{-1}(t) = \hat{V}_f^{-1}(t) \tag{126}$$

where use has been made of the unitarity of $V_f(t)$.

Equation (122) requires that

$$\langle 1_{\mathrm{tot}}|0_f(t)\rangle = 1 \tag{127}$$

where the thermal bra-vacuum $\langle 1_{\text{tot}}|$ is defined by

$$\langle 1_{\text{tot}}| \equiv \langle|\langle 1| \tag{128}$$

with the thermal bra-vacuum $\langle 1|$ in the space $\mathcal{H}_S$ of the relevant system and the thermal bra-vacuum $\langle|$ in the space Γ^β of the irrelevant system. The expectation value $\text{tr}_{\text{tot}} A\rho_f(t)$ is expressed as the expectation with respect to the thermal ket-vacuum $|0_f(t)\rangle$ and the thermal bra-vacuum $\langle 1_{\text{tot}}|$, i.e.

$$\text{tr}_{\text{tot}} A\rho_f(t) = \langle 1_{\text{tot}}|A|0_f(t)\rangle. \tag{129}$$

Note that, for any relevant system operator A, we have

$$\langle 1|A^\dagger = \langle 1|\tilde{A} \tag{130}$$

which is the basic property of thermal space [19–21]. Furthermore, for the random force operators $\mathrm{d}B_t$ and $\mathrm{d}B_t^\dagger$, we have

$$\langle|\mathrm{d}B_t^\dagger = \langle|\mathrm{d}\tilde{B}_t \tag{131}$$

which follows from (30).

The equation (127) together with (124) yields

$$\langle 1_{\text{tot}}|\hat{V}_f(t)|0_f(0)\rangle = 1. \tag{132}$$

Since equation (132) should hold for any time t and for any initial thermal vacuum $|0_f(0)\rangle$, we have

$$\langle 1_{\text{tot}}|\hat{V}_f(t) = \langle 1_{\text{tot}}|\hat{V}_f(0) = \langle 1_{\text{tot}}| \tag{133}$$

where we have used the fact that $\hat{V}_f(0) = 1$.

5.2. Stochastic Liouville equation

5.2.1. Ito type. Using the calculus rule of Ito type, we have from (125)

$$\mathrm{d}\hat{V}_f(t) = \mathrm{d}V_f(t) \cdot \tilde{V}_f(t) + V_f(t) \cdot \mathrm{d}\tilde{V}_f(t) + \mathrm{d}V_f(t)\,\mathrm{d}\tilde{V}_f(t). \tag{134}$$

Substituting (106) and its tilde conjugate

$$\mathrm{d}\tilde{V}_f(t) = \mathrm{i}\tilde{\mathcal{H}}_{f,t}\,\mathrm{d}t\,\tilde{V}_f(t) \tag{135}$$

into (134), we have

$$\mathrm{d}\hat{V}_f(t) = -\mathrm{i}\hat{\mathcal{H}}_{f,t}\,\mathrm{d}t\,\hat{V}_f(t) \tag{136}$$

where

$$\hat{\mathcal{H}}_{f,t}\,\mathrm{d}t \equiv \mathcal{H}_{f,t}\,\mathrm{d}t - \tilde{\mathcal{H}}_{f,t}\,\mathrm{d}t + \mathrm{i}\mathcal{H}_{f,t}\,\mathrm{d}t\tilde{\mathcal{H}}_{f,t}\,\mathrm{d}t. \tag{137}$$

With the help of (114) and the product rules (54), $\mathcal{H}_{f,t}\,\mathrm{d}t\tilde{\mathcal{H}}_{f,t}\,\mathrm{d}t$ is calculated as

$$\mathcal{H}_{f,t}\,\mathrm{d}t\tilde{\mathcal{H}}_{f,t}\,\mathrm{d}t = 2\kappa[(\bar{n}+1)a\tilde{a} + \bar{n}a^\dagger\tilde{a}^\dagger]\,\mathrm{d}t. \tag{138}$$

Putting (114) and (138) into (137), we obtain

$$\hat{\mathcal{H}}_{f,t}\,\mathrm{d}t = \hat{H}_S\,\mathrm{d}t + \mathrm{i}(\hat{\Pi}_{\mathrm{R}} + \hat{\Pi}_{\mathrm{D}})\,\mathrm{d}t + \mathrm{d}\hat{M}_t \tag{139}$$

where

$$\hat{H}_S = H_S - \tilde{H}_S \tag{140}$$

$$\hat{\Pi}_{\mathrm{R}} = -\kappa[(a^\dagger - \tilde{a})(\mu a + \nu\tilde{a}^\dagger) + \mathrm{TC}] \tag{141}$$

$$\hat{\Pi}_{\mathrm{D}} = 2\kappa(\bar{n}+\nu)(a^\dagger - \tilde{a})(\tilde{a}^\dagger - a) \tag{142}$$

and

$$\mathrm{d}\hat{M}_t = \mathrm{i}\{[(a^\dagger - \tilde{a})\,\mathrm{d}W_t + \mathrm{TC}] - [(\mu a + \nu\tilde{a}^\dagger)\,\mathrm{d}W_t^{\venus} + \mathrm{TC}]\} \tag{143}$$

with real numbers μ and ν satisfying $\mu + \nu = 1$. Here, TC indicates the tilde conjugate of the previous term. $\hat{\Pi}_{\boldsymbol{R}}$ and $\hat{\Pi}_{\boldsymbol{D}}$ represent relaxation and diffusion term respectively The operators $\mathrm{d}W_t$ and $\mathrm{d}W_t^{\venus}$ are defined by

$$\mathrm{d}W_t \equiv \sqrt{2\kappa}(\mu\,\mathrm{d}B_t + \nu\,\mathrm{d}\tilde{B}_t^\dagger) \qquad \mathrm{d}W_t^{\venus} \equiv \sqrt{2\kappa}(\mathrm{d}B_t^\dagger - \mathrm{d}\tilde{B}_t). \tag{144}$$

Making use of relations (130) and (131), we see that (139) satisfies

$$\langle 1_{\mathrm{tot}}|\hat{\mathcal{H}}_{f,t}\,\mathrm{d}t = \langle|\langle 1|\hat{\mathcal{H}}_{f,t}\,\mathrm{d}t = 0 \tag{145}$$

which is consistent with relation (133) and assures the conservation of probability (127). Note that

$$\langle 1|\hat{\mathcal{H}}_{f,t}\,\mathrm{d}t \neq 0 \tag{146}$$

which indicates that the conservation of probability does not hold within only the space of states of the relevant system, i.e.

$$\langle 1|0_f(t)\rangle \neq 1. \tag{147}$$

Similarly, from the definition

$$\hat{V}_f^\dagger(t) = V_f^\dagger(t)\tilde{V}_f^\dagger(t) \tag{148}$$

we obtain

$$\mathrm{d}\hat{V}_f^\dagger(t) = \mathrm{i}\hat{V}_f^\dagger(t)\hat{\mathcal{H}}_{f,t}^-\,\mathrm{d}t \tag{149}$$

with

$$\hat{\mathcal{H}}_{f,t}^-\,\mathrm{d}t = \hat{H}_S\,\mathrm{d}t - \mathrm{i}(\hat{\Pi}_{\mathrm{R}} + \hat{\Pi}_{\mathrm{D}})\,\mathrm{d}t + \mathrm{d}\hat{M}_t. \tag{150}$$

Note that $\hat{\mathcal{H}}_{f,t}^-\,\mathrm{d}t$ is not the Hermitian conjugate of $\hat{\mathcal{H}}_{f,t}\,\mathrm{d}t$.

We see that equations (136) with (139) and (149) with (150) satisfy

$$\mathrm{d}\hat{V}_f^\dagger(t)\cdot\hat{V}_f(t) + \hat{V}_f^\dagger(t)\cdot\mathrm{d}\hat{V}_f(t) + \mathrm{d}\hat{V}_f^\dagger(t)\,\mathrm{d}\hat{V}_f(t) = 0 \tag{151}$$

and

$$\mathrm{d}\hat{V}_f(t)\cdot\hat{V}_f^\dagger(t) + \hat{V}_f(t)\cdot\mathrm{d}\hat{V}_f^\dagger(t) + \mathrm{d}\hat{V}_f(t)\,\mathrm{d}\hat{V}_f^\dagger(t) = 0 \tag{152}$$

which are consistent with the unitarity of $\hat{V}_f(t)$.

Since $\hat{V}_f(t)$ and $\hat{V}_f^\dagger(t)$ are subject to the stochastic differential equations (136) with (139) and (149) with (150) of Ito type, respectively, they are quantum stochastic processes consisting of quantum stochastic integrals of Ito type. Therefore, $\hat{V}_f(t)$ and $\hat{V}_f^\dagger(t)$ are adapted processes.

Applying equation (136) of $\hat{V}_f(t)$ to the thermal vacuum $|0_f(0)\rangle$, we obtain the quantum stochastic Liouville equation of Ito type

$$\mathrm{d}|0_f(t)\rangle = -\mathrm{i}\hat{\mathcal{H}}_{f,t}\,\mathrm{d}t|0_f(t)\rangle \tag{153}$$

with the infinitesimal time-evolution generator (139).

5.2.2. Stratonovich type. Using the calculus rule of Stratonovich type, we have from (125)

$$\mathrm{d}\hat{V}_f(t) = \mathrm{d}V_f(t) \circ \tilde{V}_f(t) + V_f(t) \circ \mathrm{d}\tilde{V}_f(t). \tag{154}$$

Substituting (116) and its tilde conjugate

$$\mathrm{d}\tilde{V}_f(t) = \mathrm{i}\tilde{H}_{f,t}\,\mathrm{d}t \circ \tilde{V}_f(t) \tag{155}$$

into (154), we obtain

$$\mathrm{d}\hat{V}_f(t) = -\mathrm{i}\hat{H}_{f,t}\,\mathrm{d}t \circ \hat{V}_f(t) \tag{156}$$

where

$$\hat{H}_{f,t}\,\mathrm{d}t \equiv H_{f,t}\,\mathrm{d}t - \tilde{H}_{f,t}\,\mathrm{d}t. \tag{157}$$

Putting (118) into (157), we get

$$\hat{H}_{f,t}\,\mathrm{d}t = \hat{H}_S\,\mathrm{d}t + \mathrm{d}\hat{M}_t \tag{158}$$

which is apparently Hermitian.

Similarly, from definition (148), we obtain the equation of $\hat{V}_f^\dagger(t)$ as

$$\mathrm{d}\hat{V}_f^\dagger(t) = \mathrm{i}\hat{V}_f^\dagger(t) \circ \hat{H}_{f,t}\,\mathrm{d}t \tag{159}$$

where use has been made of the hermiticy of $\hat{H}_{f,t}\,\mathrm{d}t$. Note that (159) is the Hermitian conjugate of equation (156).

Equations (156) and (159) with (158) satisfy the following equations,

$$\mathrm{d}\hat{V}_f^\dagger(t) \circ \hat{V}_f(t) + \hat{V}_f^\dagger(t) \circ \mathrm{d}\hat{V}_f(t) = 0 \tag{160}$$

$$\mathrm{d}\hat{V}_f(t) \circ \hat{V}_f^\dagger(t) + \hat{V}_f(t) \circ \mathrm{d}\hat{V}_f^\dagger(t) = 0 \tag{161}$$

which show the unitarity of $\hat{V}_f(t)$.

With the help of properties (130) and (131), we see that expression (158) satisfies

$$\langle 1_{\mathrm{tot}}|\hat{H}_{f,t}\,\mathrm{d}t = \langle|\langle 1|\hat{H}_{f,t}\,\mathrm{d}t = 0 \tag{162}$$

which is consistent with (133).

The time-evolution equation (136) of Ito type with (39) is connected to equation (156) of Stratonovich type with (158) by the relation (89) between the Ito and the Stratonovich products. In the same way, equation (149) of Ito type with (150) is connected to equation (159) of Stratonovich type with (158) through the relation (88).

Applying equation (156) of $\hat{V}_f(t)$ to the thermal vacuum $|0_f(0)\rangle$, we have the quantum stochastic Liouville equation of Stratonovich type

$$\mathrm{d}|0_f(t)\rangle = -\mathrm{i}\hat{H}_{f,t}\,\mathrm{d}t \circ |0_f(t)\rangle \tag{163}$$

with the infinitesimal time-evolution generator (158).

5.3. Quantum master equation

Applying the stochastic Liouville equation (153) of Ito type with the infinitesimal time-evolution generator (139) to the random force bra-vacuum $\langle|$, we have

$$\begin{aligned}\mathrm{d}\langle|0_f(t)\rangle &= -\mathrm{i}\langle|\hat{\mathcal{H}}_{f,t}\,\mathrm{d}t|0_f(t)\rangle \\ &= -\mathrm{i}[\{\hat{H}_S + \mathrm{i}(\hat{\Pi}_{\mathrm{R}} + \hat{\Pi}_{\mathrm{D}})\}\,\mathrm{d}t\langle|0_f(t)\rangle + \langle|\mathrm{d}\hat{M}_t|0_f(t)\rangle].\end{aligned} \tag{164}$$

Under the assumption†

$$|0_f(0)\rangle = |0_S\rangle|\rangle \tag{165}$$

with the thermal vacuum $|0_S\rangle$ of the relevant system at $t = 0$, $\langle|\mathrm{d}\hat{M}_t|0_f(t)\rangle$ can be evaluated as

$$\langle|\mathrm{d}\hat{M}_t|0_f(t)\rangle = \langle|\mathrm{d}\hat{M}_t\hat{V}_f(t)|\rangle|0_S\rangle = 0 \tag{166}$$

where we have used the definition (124) of the thermal vacuum $|0_f(t)\rangle$ and the property (83) of the products of Ito type. Therefore, putting $|0(t)\rangle = \langle|0_f(t)\rangle$, we finally obtain the quantum master equation for the bi-linear and phase invariant system as

$$\frac{\partial}{\partial t}|0(t)\rangle = -\mathrm{i}\hat{H}|0(t)\rangle \tag{167}$$

where the infinitesimal time-evolution generator $\hat{H}$ is given by

$$\hat{H} = \hat{H}_S + \mathrm{i}\hat{\Pi} \tag{168}$$

with

$$\begin{aligned}\hat{\Pi} &= \hat{\Pi}_{\mathrm{R}} + \hat{\Pi}_{\mathrm{D}} \\ &= -\kappa[(1+2\bar{n})(a^\dagger a + \tilde{a}^\dagger\tilde{a}) - 2(1+\bar{n})a\tilde{a} - 2\bar{n}a^\dagger\tilde{a}^\dagger] - 2\kappa\bar{n}\end{aligned} \tag{169}$$

which is identical to that obtained within the framework of NETFD [17–21]. Note that we can also derive the quantum master equation (167) by applying the stochastic Liouville equation (163) of Stratonovich type with the infinitesimal time-evolution generator (158) to the random force bra-vacuum $\langle|$ [19, 22].

Recalling equation (124) and taking (165) into account, we find that

$$|0(t)\rangle = \langle|\hat{V}_f(t)|\rangle|0_S\rangle. \tag{170}$$

On the other hand, the time-evolution generator $\hat{V}(t)$ of the thermal ket-vacuum $|0(t)\rangle$ is defined by

$$|0(t)\rangle = \hat{V}(t)|0(0)\rangle. \tag{171}$$

Provided that $|0(0)\rangle = |0_S\rangle$, equations (170) and (171) yield

$$\hat{V}(t) = \langle|\hat{V}_f(t)|\rangle. \tag{172}$$

5.4. *Quantum Langevin equation*

5.4.1. Operators in the Heisenberg representation. Since $\hat{V}_f(t)$ is unitary, we have

$$\hat{V}_f^\dagger(t)\hat{V}_f(t) = \hat{V}_f(t)\hat{V}_f^\dagger(t) = 1. \tag{173}$$

From equation (173), we see with the help of property (133) that

$$\langle 1_{\mathrm{tot}}|\hat{V}_f^\dagger(t) = \langle 1_{\mathrm{tot}}|. \tag{174}$$

The expectation value of any observable A with respect to the state $|0_f(t)\rangle$ is given by

$$\langle 1_{\mathrm{tot}}|A|0_f(t)\rangle = \langle 1_{\mathrm{tot}}|A\hat{V}_f(t)|0_f(0)\rangle = \langle 1_{\mathrm{tot}}|\hat{V}_f^\dagger(t)A\hat{V}_f(t)|0_f(0)\rangle \tag{175}$$

where we have used equation (124) and property (174). If we define the operator in the Heisenberg representation

$$A(t) = \hat{V}_f^\dagger(t)A\hat{V}_f(t) \tag{176}$$

† Equation (165) indicates that $|0_f(0)\rangle$ is the tensor product of $|0_S\rangle$ and $|\rangle$, i.e. $|0_f(0)\rangle = |0_S\rangle \otimes |\rangle$. In the following, we omit the symbol $\otimes$.

we consider (175) to be the expectation value of $A(t)$ with respect to the initial state $|0_f(0)\rangle$. Note that, as $\hat{V}_f(t)$ and $\hat{V}_f^\dagger(t)$ are adapted processes, the operator $A(t)$ defined by (176) is also an adapted process. Therefore, the following commutation relation holds

$$[A(t), \mathrm{d}B_t] = [A(t), \mathrm{d}B_t^\dagger] = [A(t), \mathrm{d}\tilde{B}_t] = [A(t), \mathrm{d}\tilde{B}_t^\dagger] = 0 \tag{177}$$

for quantum Wiener processes B_t, $B_t^\dagger$ and their tilde conjugates $\tilde{B}_t$, $\tilde{B}_t^\dagger$, which comes from (80).

Any operators in the Heisenberg representation defined by (176) keep the equal-time commutation relations, such as

$$[a(t), a^\dagger(t)] = 1 \qquad [\tilde{a}(t), \tilde{a}^\dagger(t)] = 1. \tag{178}$$

Note that, using the properties (130) and (174), we have for $A(t)$ defined by (176)

$$\langle 1_{\mathrm{tot}}|A^\dagger(t) = \langle 1_{\mathrm{tot}}|\tilde{A}(t). \tag{179}$$

5.4.2. Ito type. Using the calculus rule of Ito type, we have the algebraic identity for the operator $A(t)$ defined by (176)

$$\mathrm{d}A(t) = \mathrm{d}\hat{V}_f^\dagger(t) A \hat{V}_f(t) + \hat{V}_f^\dagger(t) A\, \mathrm{d}\hat{V}_f(t) + \mathrm{d}\hat{V}_f^\dagger(t) A\, \mathrm{d}\hat{V}_f(t). \tag{180}$$

Substituting equations (136) with (139) and (149) with (150) into (180), we obtain the quantum Langevin equation of Ito type:

$$\begin{aligned}
\mathrm{d}A(t) = \mathrm{i}[\hat{H}_S(t), A(t)]\,\mathrm{d}t &+ \kappa\{[a^\dagger(t) - \tilde{a}(t), A(t)](\mu a(t) + \nu\tilde{a}^\dagger(t)) \\
&- (a^\dagger(t) - \tilde{a}(t))[\mu a(t) + \nu\tilde{a}^\dagger(t), A(t)] \\
&+ [\tilde{a}^\dagger(t) - a(t), A(t)](\mu\tilde{a}(t) + \nu a^\dagger(t)) \\
&- (\tilde{a}^\dagger(t) - a(t))[\mu\tilde{a}(t) + \nu a^\dagger(t), A(t)]\}\,\mathrm{d}t \\
&+ 2\kappa(\bar{n} + \nu)[\tilde{a}^\dagger(t) - a(t), [a^\dagger(t) - \tilde{a}(t), A(t)]]\,\mathrm{d}t \\
&- \{[a^\dagger(t) - \tilde{a}(t), A(t)]\,\mathrm{d}W_t + [\tilde{a}^\dagger(t) - a(t), A(t)]\,\mathrm{d}\tilde{W}_t\} \\
&+ \{[\mu a(t) + \nu\tilde{a}^\dagger(t), A(t)]\,\mathrm{d}W_t^{\natural} + [\mu\tilde{a}(t) + \nu a^\dagger(t), A(t)]\,\mathrm{d}\tilde{W}_t^{\natural}\}.
\end{aligned} \tag{181}$$

5.4.3. Stratonovich type. Making use of the calculus rule of Stratonovich type, we have the algebraic identity for the operator $A(t)$ defined by (176)

$$\mathrm{d}A(t) = \mathrm{d}\hat{V}_f^\dagger(t) \circ A\hat{V}_f(t) + \hat{V}_f^\dagger(t) A \circ \mathrm{d}\hat{V}_f(t). \tag{182}$$

With the help of equation (156) and its Hermitian conjugate (159) together with the identity (182), we obtain the quantum Langevin equation of Stratonovich type. We see that the quantum Langevin equation of Stratonovich type can be expressed as the Heisenberg equation for $A(t)$:

$$\mathrm{d}A(t) = \mathrm{i}[\hat{H}_f(t)\,\mathrm{d}t \,\natural\, A(t)] \tag{183}$$

where we have defined

$$\hat{H}_f(t)\,\mathrm{d}t = \hat{V}_f^\dagger(t) \circ \hat{H}_{f,t}\,\mathrm{d}t \circ \hat{V}_f(t). \tag{184}$$

The symbol $[\cdot \natural \cdot]$ is the commutator defined by (84). Recalling (158), we have the explicit form of equation (183) as

$$\begin{aligned}
\mathrm{d}A(t) = \mathrm{i}[\hat{H}_S(t), A(t)]\,\mathrm{d}t &\\
&- \{[a^\dagger(t) - \tilde{a}(t), A(t)] \circ \mathrm{d}W(t) + [\tilde{a}^\dagger(t) - a(t), A(t)] \circ \mathrm{d}\tilde{W}(t)\} \\
&+ \{[\mu a(t) + \nu\tilde{a}^\dagger(t), A(t)] \circ \mathrm{d}W^{\natural}(t) \\
&+ [\mu\tilde{a}(t) + \nu a^\dagger(t), A(t)] \circ \mathrm{d}\tilde{W}^{\natural}(t)\}
\end{aligned} \tag{185}$$

where we defined the operators $\mathrm{d}W(t)$, $\mathrm{d}W^{\natural}(t)$, $\mathrm{d}\tilde{W}(t)$, $\mathrm{d}\tilde{W}^{\natural}(t)$ by

$$\mathrm{d}W(t) \equiv \hat{V}_f^{\dagger}(t) \circ \mathrm{d}W_t \circ \hat{V}_f(t) \tag{186}$$

$$\mathrm{d}W^{\natural}(t) \equiv \hat{V}_f^{\dagger}(t) \circ \mathrm{d}W_t^{\natural} \circ \hat{S}_f(t) \tag{187}$$

$$\mathrm{d}\tilde{W}(t) \equiv \hat{V}_f^{\dagger}(t) \circ \mathrm{d}\tilde{W}_t \circ \hat{V}_f(t) \tag{188}$$

$$\mathrm{d}\tilde{W}^{\natural}(t) \equiv \hat{V}_f^{\dagger}(t) \circ \mathrm{d}\tilde{W}_t^{\natural} \circ \hat{S}_f(t). \tag{189}$$

Using the relations (88) and (89) between the products of Ito and Stratonovich types and the product rules (54), we can express $\mathrm{d}W(t)$, $\mathrm{d}W^{\natural}(t)$, $\mathrm{d}\tilde{W}(t)$, $\mathrm{d}\tilde{W}^{\natural}(t)$ in terms of $\mathrm{d}W_t$, $\mathrm{d}W_t^{\natural}$, $\mathrm{d}\tilde{W}_t$, $\mathrm{d}\tilde{W}_t^{\natural}$, respectively, as follows:

$$\mathrm{d}W(t) = \mathrm{d}W_t - \kappa[\mu a(t) + \nu\tilde{a}^{\dagger}(t)]\,\mathrm{d}t \tag{190}$$

$$\mathrm{d}W^{\natural}(t) = \mathrm{d}W_t^{\natural} - \kappa[a^{\dagger}(t) - \tilde{a}(t)]\,\mathrm{d}t \tag{191}$$

$$\mathrm{d}\tilde{W}(t) = \mathrm{d}\tilde{W}_t - \kappa[\mu\tilde{a}(t) + \nu a^{\dagger}(t)]\,\mathrm{d}t \tag{192}$$

$$\mathrm{d}\tilde{W}^{\natural}(t) = \mathrm{d}\tilde{W}_t^{\natural} - \kappa[\tilde{a}^{\dagger}(t) - a(t)]\,\mathrm{d}t. \tag{193}$$

Substituting (190)–(193) into (185), we see that the quantum Langevin equation (185) of Stratonovich type becomes

$$\begin{aligned}\mathrm{d}A(t) = {} & \mathrm{i}[\hat{H}_S(t), A(t)]\,\mathrm{d}t \\ & +\kappa\{[a^{\dagger}(t) - \tilde{a}(t), A(t)](\mu a(t) + \nu\tilde{a}^{\dagger}(t)) \\ & -(a^{\dagger}(t) - \tilde{a}(t))[\mu a(t) + \nu\tilde{a}^{\dagger}(t), A(t)] \\ & +[\tilde{a}^{\dagger}(t) - a(t), A(t)](\mu\tilde{a}(t) + \nu a^{\dagger}(t)) \\ & -(\tilde{a}^{\dagger}(t) - a(t))[\mu\tilde{a}(t) + \nu a^{\dagger}(t), A(t)]\}\,\mathrm{d}t \\ & -\{[a^{\dagger}(t) - \tilde{a}(t), A(t)] \circ \mathrm{d}W_t + [\tilde{a}^{\dagger}(t) - a(t), A(t)] \circ \mathrm{d}\tilde{W}_t\} \\ & +\{\mathrm{d}W_t^{\natural} \circ [\mu a(t) + \nu\tilde{a}^{\dagger}(t), A(t)] \\ & +\mathrm{d}\tilde{W}_t^{\natural} \circ [\mu\tilde{a}(t) + \nu a^{\dagger}(t), A(t)]\}.\end{aligned} \tag{194}$$

Furthermore, with the help of the relations (88) and (89) between the products of Ito and Stratonovich types and the product rules (54), we find that the equation of Stratonovich type (194) is identical to equation (181) of Ito type.

5.5. The equation of motion of the expectation value

Let us assume that the initial vacuum $|0_f(0)\rangle \equiv |0_f\rangle$ can be expressed by the product of the vacuums of the relevant and the irrelevant systems as (165).

Applying the quantum Langevin equation (181) of Ito type to the bra-vacuum $\langle 1_{\mathrm{tot}}|$, we have

$$\begin{aligned}\mathrm{d}\langle 1_{\mathrm{tot}}|A(t) = {} & \mathrm{i}\langle 1_{\mathrm{tot}}|[H_S(t), A(t)]\,\mathrm{d}t \\ & +\kappa(\langle 1_{\mathrm{tot}}|a^{\dagger}(t)[A(t), a(t)] + \langle 1_{\mathrm{tot}}|[a^{\dagger}(t), A(t)]a(t))\,\mathrm{d}t \\ & +2\kappa\bar{n}\langle 1_{\mathrm{tot}}|[a^{\dagger}(t), [A(t), a(t)]]\,\mathrm{d}t \\ & +\langle 1_{\mathrm{tot}}|[A(t), a^{\dagger}(t)]\,\mathrm{d}F_t + \langle 1_{\mathrm{tot}}|[a(t), A(t)]\,\mathrm{d}F_t^{\dagger}\end{aligned} \tag{195}$$

where we have used properties (131) and (179).

Putting the ket-vacuum $|0_f\rangle$ into (195), we obtain the equation of motion of the expectation value of an arbitrary operator A of the relevant system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle 1_{\mathrm{tot}}|A(t)|0_f\rangle = \mathrm{i}\langle 1_{\mathrm{tot}}|[H_S(t), A(t)]|0_f\rangle$$

$$+\kappa(\langle 1_{\rm tot}|a^\dagger(t)[A(t),a(t)]|0_f\rangle+\langle 1_{\rm tot}|[a^\dagger(t),A(t)]a(t)|0_f\rangle)$$
$$+2\kappa\bar{n}\langle 1_{\rm tot}|[a^\dagger(t),[A(t),a(t)]]|0_f\rangle. \qquad (196)$$

Here, we have used the property (83) of the Ito products.

Remembering (172) and the definition (176) of $A(t)$, we find with the assumption (165) that

$$\langle 1_{\rm tot}|A(t)|0_f\rangle=\langle|\langle 1|\hat{V}_f^\dagger(t)A\hat{V}_f(t)|0_S\rangle|\rangle=\langle 1|A|0(t)\rangle \qquad (197)$$

where we have used property (174) and the assumption that $|0(0)\rangle=|0_S\rangle$. Taking account of relation (197), we see that equation (196) of the expectation value is identical to the equation derived from the master equation (167) with (168) and (169), which shows the consistency of the framework.

6. Summary and discussion

In this paper, we constructed the quantum Wiener processes together with their representation space by extending the work of mathematicians and by implanting it into NETFD. Then, we constructed a unified system of quantum stochastic differential equations on the basis of the stochastic Schrödinger equation which was studied by mathematicians.

The quantum Wiener processes were constructed by using boson operators with time indices. When we adopted the Fock space Γ^0 for the representation space, in the same way as Hudson and Parthasarathy, we obtained the quantum Wiener processes at zero temperature. However, we obtained the quantum Wiener processes at finite temperatures by extending the representation space to the Fock space Γ^β which is obtained by the Bogoliubov transformation in the tensor product space $\Gamma=\Gamma^0\otimes\tilde{\Gamma}^0$. This is the reconstruction of quantum Wiener processes at finite temperatures introduced by Hudson and Lindsay [7, 8], within the framework of NETFD. Within NETFD, the thermal degree of freedom was introduced by the thermal state conditions or the Bogoliubov transformation, which is a manifestation of unitary inequivalence between the thermal vacuums of zero and finite temperatures. This notion of unitary inequivalence between the vacuums with different temperatures is one of the remarkable features within NETFD or TFD. The quantum Wiener processes and the quantum stochastic calculus given in this paper provide the foundation for those used in quantum optics [3] and quantum stochastic differential equations within NETFD [22–32].

We constructed the stochastic Schrödinger equation with the quantum Wiener processes at finite temperatures on the requirement that the time-evolution generator should be unitary. Then, we introduced the density operator corresponding to the stochastic wavefunction. By means of the principle of correspondence between quantities in thermal space and in Hilbert space, we obtained the stochastic thermal ket-vacuum corresponding to the density operator. The time-evolution equation (Schrödinger equation) of the thermal ket-vacuum gave the quantum stochastic Liouville equation. On the other hand, the Heisenberg equation with the infinitesimal time-evolution generator of the quantum stochastic Liouville equation gave the quantum Langevin equation. Using the quantum stochastic calculus constructed in section 3, we constructed the quantum stochastic differential equations both of Ito and of Stratonovich types.

Applying the stochastic Liouville equation of Ito type to the random force bra-vacuum $\langle|$, we obtained the quantum master equation, which is identical to that derived in papers [17–21]. Taking the expectation with respect to the thermal ket-vacuum $|0_f\rangle$ and the thermal bra-vacuum $\langle 1_{\rm tot}|$ of the quantum Langevin equation of Ito type, we obtained the equation of motion of the expectation value of an arbitrary relevant system operator. This equation of motion is equivalent to that derived by the master equation, which shows the self-consistency of the system.

Hudson and Lindsay constructed a unitary stochastic time evolution in the vector space $\mathcal{H}_S^0 \otimes \Gamma^\beta$, where $\mathcal{H}_S^0$ for the relevant system is a usual Hilbert space and Γ^β for random force operators is a Fock space in thermal space, which was briefly reviewed in section 4. The fact that the vector space for the relevant system is not a thermal space prevents the system from introducing the quantum stochastic Liouville equation. In this paper, we gave the quantum stochastic Liouville equation by adopting a thermal space for the space of states of the relevant system as well as for the random force system.

The stochastic time evolution in thermal space constructed in this paper is unitary. On the other hand, non-unitary stochastic time evolution was constructed within the framework of NETFD [22–32]. In this way, it turned out that there exist two kinds of systems in quantum stochastic differential equations within NETFD; one is the system of non-unitary stochastic time evolution and the other is that of unitary stochastic time evolution. The two systems are equivalent in the sense that they give the same equation of motion of the expectation value of any observable. The relation between the two systems will be investigated in a forthcoming paper.

Appendix. The principle of correspondence

The correspondence between vectors in the thermal space and operators in a Hilbert space is given by the following rule [17, 18, 34]:

$$\rho_S(t) \longleftrightarrow |0(t)\rangle \tag{198}$$

$$A_1\rho_S(t)A_2 \longleftrightarrow A_1\tilde{A}_2^\dagger|0(t)\rangle. \tag{199}$$

Here, $\rho_S(t)$ is a density operator on the Hilbert space, whereas $|0(t)\rangle$ is a thermal ket-vacuum in the thermal space. A_1 and A_2 are arbitrary operators on the Hilbert space.

It was noticed first by Crawford [35] that the introduction of two kinds of operators for each operator enables us to handle the Liouville equation as the Schrödinger equation.

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